Chapter 2: Force Vectors

Chapter Objectives

• To show how to add forces and resolve them into components using the Parallelogram Law.

• To express force and position in Cartesian vector form and explain how to determine the vector’s magnitude and direction.

• To introduce the Dot Product in order to determine the angle between two vectors or the projection of one vector onto another.

2.1 Scalars and Vectors

Scalar. A “scalar” is any physical quantity that can be completely specified by its magnitude (e.g. length, time, mass, speed, temperature).

Vector. A “vector” is any physical quantity that requires both magnitude and direction for its complete description (e.g. force, velocity, acceleration).

A vector may be represented by boldfaced letters (as in a textbook), by underlining the letter (\( \mathbf{P} \)), or by drawing a short arrow above the letter.

Graphically, a vector can be represented by a line segment.

• Through the use of an appropriate scale, the length of the line may be chosen to represent the magnitude of the vector, such as in the case of a force (e.g. 1 inch = 5 lb).

The sense, or direction, of the force may be indicated by an arrowhead and by an angle of inclination measured from a reference (such as the horizontal) plane.

2.2 Vector Operations

Multiplication and Division of a Vector by a Scalar
The product of a scalar “a” and a vector \( \mathbf{A} \) is a vector written as \( a\mathbf{A} \) and having a magnitude \( aA \).

Division of a vector by a scalar can be defined using the laws of multiplication since \( \mathbf{A}/a = (1/a) \mathbf{A} \), where \( a \neq 0 \).
Vector Addition
Two vectors \( \mathbf{A} \) and \( \mathbf{B} \) such as force or position, may be added to form a "resultant" vector \( \mathbf{R} = \mathbf{A} + \mathbf{B} \) by using the "Parallelogram Law."

Experimental evidence shows that two force vectors, \( \mathbf{A} \) and \( \mathbf{B} \), acting on particle \( A \) may be replaced by a single vector \( \mathbf{R} \) that has the same effect on the particle.
- This force vector is called the "resultant" of the forces \( \mathbf{A} \) and \( \mathbf{B} \) and may be obtained by constructing a parallelogram, using \( \mathbf{A} \) and \( \mathbf{B} \) as the sides of the parallelogram.
- The diagonal that passes through \( A \) represents the resultant.
- The "Parallelogram Law" is based on experimental evidence and cannot be proven or derived mathematically.

The magnitude of \( \mathbf{A} + \mathbf{B} \) is **not** equal to the algebraic summation of the magnitudes \( (\mathbf{A} + \mathbf{B}) \) of the vectors \( \mathbf{A} \) and \( \mathbf{B} \), except in the special case where \( \mathbf{A} \) and \( \mathbf{B} \) are collinear.
- When \( \mathbf{A} \) and \( \mathbf{B} \) are collinear, the Parallelogram Law reduces to an algebraic or scalar addition \( \mathbf{R} = \mathbf{A} + \mathbf{B} \).

Since the Parallelogram that is constructed from the vectors \( \mathbf{A} \) and \( \mathbf{B} \) does not depend upon the order in which \( \mathbf{A} \) and \( \mathbf{B} \) are selected, the addition of two vectors is "commutative," that is
\[
\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}
\]
An alternative method for determining the sum of two vectors is the “triangle rule.”
• Since the side of the parallelogram opposite to $\mathbf{B}$ is equal to $\mathbf{B}$ in magnitude and direction, we need only draw half of the parallelogram.
• Arrange $\mathbf{A}$ and $\mathbf{B}$ in a “tip-to-tail” fashion by connecting the tail of $\mathbf{A}$ with the tip of $\mathbf{B}$.
• Or, since the addition of vectors is commutative, arrange $\mathbf{A}$ and $\mathbf{B}$ in a “tip-to-tail” fashion by connecting the tail of $\mathbf{B}$ with the tip of $\mathbf{A}$.

Vector Subtraction
The subtraction of a vector is defined as the addition of the corresponding negative vector.

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

A “negative” vector has the same magnitude as another but is opposite in sense or direction (“equal and opposite”).

2.3 Vector Addition of Forces
Two common problems in statics involve either finding the resultant force, knowing its components, or resolving a known force into two or three components.

Finding a Resultant Force
The sum of three vectors is obtained by first adding the vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ and then adding the vector $\mathbf{F}_3$ to the vector resultant of $\mathbf{F}_1 + \mathbf{F}_2$.

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (\mathbf{F}_1 + \mathbf{F}_2) + \mathbf{F}_3$$

The sum of any number of vectors may be obtained by applying repeatedly the Parallelogram Law (or Triangle Rule) to successive pairs of vectors until all of the given vectors are replaced by a single vector.
An alternate method (similar to the Triangle Rule for two vectors) is the “Polygon Rule” for the addition of more than two vectors.

- Arrange the given vectors in a “tip-to-tail” fashion and connect the tail of the first vector with the tip of the last one to form the single resultant vector.

Vector addition is “associative.”

\[ \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (\mathbf{F}_1 + \mathbf{F}_2) + \mathbf{F}_3 = \mathbf{F}_1 + (\mathbf{F}_2 + \mathbf{F}_3) \]

Consider particle A acted upon by several coplanar forces.

- Since the forces all pass through A, they are said to be "concurrent."
- The vectors representing the forces acting on A may be added according to the Polygon Rule.

The vector \( \mathbf{R} \) represents the resultant of the given concurrent forces.

**Finding the Components of a Force**

A vector \( \mathbf{R} \) may be replaced by “components.”

- This process is known as resolving the force \( \mathbf{R} \) into components.
- For each force \( \mathbf{R} \) there is an infinite number of possible sets of component forces.
Two cases are of particular interest.  
1. One of the two components of \( \mathbf{R} \) is known.  
   - The second component \( \mathbf{B} \) is obtained by applying the Triangle Rule and joining the tip of \( \mathbf{A} \) to the tip of \( \mathbf{R} \).
   - The magnitude and direction of \( \mathbf{B} \) are determined graphically or by trigonometry using the Law of Sines and the Law of Cosines.

   Law of Sines: \( \frac{\sin A}{a} = \frac{\sin B}{b} \)

   Law of Cosines: \( b^2 = a^2 + c^2 - 2ac \cos B \)

2. The line of action of each component is known.  
   - The magnitude and sense of the components are found by applying the Parallelogram Law and drawing lines.
   - The magnitudes of the components may be determined graphically or by trigonometry.

Most commonly we will use the rectangular components of a force, that is, the components that are parallel to the x- and y-axes.

**Addition of Several Forces**

As previously demonstrated, if more than two forces are to be added, successive applications of the parallelogram law can be performed to obtain the resultant force.

**Miscellaneous**

The operations of vector addition and the product of a scalar and a vector involve the following principles.
   - The product is associative with respect to scalar multiplication.
     \[ a (b \mathbf{U}) = a b (\mathbf{U}) \]
• The product is distributive with respect to scalar addition.
  \[(a + b) \mathbf{U} = a \mathbf{U} + b \mathbf{U}\]

• The product is distributive with respect to vector addition.
  \[a ( \mathbf{U} + \mathbf{V} ) = a \mathbf{U} + a \mathbf{V}\]

A force acting on a given particle may have a well-defined point of application, namely, the particle itself.

• Such a vector is said to be "fixed", or "bound" and cannot be moved without changing the conditions of the problem.

Other physical quantities, such as couples (discussed in Chapter 4), are also represented by vectors and may be freely moved in space ("free vectors").

Still other physical quantities, such as forces acting on a rigid body, are represented by vectors which may be moved, or slid, along their lines of action ("sliding vectors").

Two vectors that have the same magnitude and the same direction are "equal", whether or not they have the same point of application.
Example Problem - Resultant Force

Given: The two forces shown.

Find: Magnitude and direction of the resultant force
   a) graphically
   b) by trigonometry

a) Graphical solution.

Let 1” = 200 lb

\[ R = 906.3 \text{ lb} \]
b) By trigonometry

\[ R^2 = (800)^2 + (500)^2 - 2 \times 800 \times 500 \cos 85^\circ \]
\[ R = 905.7 \text{ lb} \]

\[ \frac{\sin \varphi}{500} = \frac{\sin 85^\circ}{905.7} \]
\[ \sin \varphi = 0.550 \]
\[ \varphi = 33.4^\circ \]

\[ \theta = 60^\circ - \varphi \]
\[ = 60^\circ - 33.4^\circ \]
\[ \theta = 26.6^\circ \]

\[ R = 905.7 \text{ lb} \]
2.4 Addition of a System of Coplanar Forces
In many problems, it will be found desirable to resolve a force into two components that are perpendicular to each other.

The force \( \mathbf{F} \) has been resolved into a horizontal component \( \mathbf{F}_x \) along the x-axis and a vertical component \( \mathbf{F}_y \) along the y-axis.

- The parallelogram that is drawn to obtain the two components is a rectangle.
- \( \mathbf{F}_x \) and \( \mathbf{F}_y \) are called "rectangular components" and
  \[ \mathbf{F} = \mathbf{F}_x + \mathbf{F}_y \]

The parallelogram becomes a rectangle.

Scalar Notation
The magnitudes of the rectangular components of \( \mathbf{F} \) can be expressed as algebraic scalars \( F_x \) and \( F_y \).

- Using \( F \) to denote the magnitude of the force and \( \theta \) the angle between \( \mathbf{F} \) and the x-axis, the scalar components of \( \mathbf{F} \) can be determined by the following.
  \[ F_x = F \cos \theta \quad \text{and} \quad F_y = F \sin \theta \]

  Note: The force triangle is "similar" geometrically to the dimensional triangle.

Cartesian Vector Notation
Two vectors of magnitude 1, directed along the positive x- and y-axes are called "unit vectors" \( \mathbf{i} \) and \( \mathbf{j} \), respectively.

The rectangular components \( \mathbf{F}_x \) and \( \mathbf{F}_y \) of a force \( \mathbf{F} \) may be obtained by multiplying respectively the unit vectors \( \mathbf{i} \) and \( \mathbf{j} \) by appropriate scalar values.

  \[ \mathbf{F}_x = F_x \mathbf{i} \quad \text{and} \quad \mathbf{F}_y = F_y \mathbf{j} \]

  and \( \mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} \)

The scalars \( F_x \) and \( F_y \) may be positive or negative.

- The absolute values are equal to the magnitudes of the rectangular components \( \mathbf{F}_x \) and \( \mathbf{F}_y \) of the force \( \mathbf{F} \).

Coplanar Force Resultants
Either of the two methods just described (i.e. the graphical method and the method using trigonometry) can be used to determine the resultant of several coplanar forces.
A third method of finding a resultant force uses the components of the given forces.

- Each force is represented as a *Cartesian vector* by resolving each force into its rectangular components.

\[ \mathbf{F}_1 = F_{1x} \mathbf{i} + F_{1y} \mathbf{j} \]
\[ \mathbf{F}_2 = F_{2x} \mathbf{i} + F_{2y} \mathbf{j} \]
\[ \mathbf{F}_3 = F_{3x} \mathbf{i} + F_{3y} \mathbf{j} \]

- The vector resultant can be determined as follows.

\[ \mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \]
\[ = F_{1x} \mathbf{i} + F_{1y} \mathbf{j} + F_{2x} \mathbf{i} + F_{2y} \mathbf{j} + F_{3x} \mathbf{i} + F_{3y} \mathbf{j} \]
\[ = (F_{1x} + F_{2x} + F_{3x}) \mathbf{i} + (F_{1y} + F_{2y} + F_{3y}) \mathbf{j} \]
\[ \mathbf{F}_R = (F_{Rx}) \mathbf{i} + (F_{Ry}) \mathbf{j} \]

- If a *scalar notation* is used, the magnitudes of the components of the resultant force can be determined as follows.

\[ F_{Rx} = F_{1x} + F_{2x} + F_{3x} \]
\[ F_{Ry} = F_{1y} + F_{2y} + F_{3y} \]

- In general, the magnitudes of the components of the resultant force can be determined as follows.

\[ F_{Rx} = \sum F_x \text{ and } F_{Ry} = \sum F_y \]

Thus, the scalar components \( F_{Rx} \) and \( F_{Ry} \) of the resultant \( \mathbf{F}_R \) of several forces acting on a particle are obtained by adding algebraically the corresponding scalar components of the given forces.

- The magnitude of \( \mathbf{F} \) is given in terms of its components by the Pythagorean Theorem.

\[ |\mathbf{F}_R| = (F_{Rx}^2 + F_{Ry}^2)^{\frac{1}{2}} \]

With this equation, you can determine the magnitude of the vector when you know its components.

- The direction angle \( \theta \), which specifies the direction of the force, is determined from trigonometry.

\[ \tan \theta = F_{Ry}/F_{Rx} \text{ and } \theta = \tan^{-1}(F_{Ry}/F_{Rx}) \]
Example Problem - Resultant Force

Note: This problem was solved previously using the graphical method and by trigonometry.
- Now the problem will be solved using the “analytical” solution.

Given: The two forces shown.

Find: The resultant force.

\[ P = 800 \text{ lb} \]
\[ P_x = 800 \cos 60^\circ = 400.0 \text{ lb} \rightarrow \]
\[ P_y = 800 \sin 60^\circ = 692.8 \text{ lb} \uparrow \]

\[ Q = 500 \text{ lb} \]
\[ Q_x = 500 \cos 35^\circ = 409.6 \text{ lb} \rightarrow \]
\[ Q_y = - 500 \sin 35^\circ = -286.8 \text{ lb} \]
\[ Q_y = 286.8 \text{ lb} \downarrow \]

The components of the resultant are determined as follows.

\[ R_x = 800 \cos 60^\circ + 500 \cos 35^\circ = 400.0 + 409.6 = 809.6 \text{ lb} \]
\[ R_y = 800 \sin 60^\circ - 500 \sin 35^\circ = 692.8 - 286.8 = 406.0 \text{ lb} \]

\[ R = [(809.6)^2 + (406.0)^2]^{\frac{1}{2}} = 905.7 \text{ lb} \]

\[ \tan \theta = 406.0/809.6 = 0.501 \quad \text{and} \quad \theta = 26.6^\circ \]

Thus, the resultant \( R = 905.7 \text{ lb} \)

Compare with previous answers:
- Graphical method: \( R = 906.3 \text{ lb}, \theta = 27^\circ \)
- Trigonometric method: \( R = 905.7, \theta = 26.6^\circ \)
2.5 Cartesian Vectors

Right-Handed Coordinate System
A rectangular or Cartesian coordinate system is said to be right-handed if the thumb of the right hand points in the direction of the positive z-axis when the fingers of the right hand are curled about this axis when directed from the positive x towards the positive y-axis.

Rectangular Components of a Vector
A vector may have one, two, or three rectangular components along the x, y, z coordinate axes.

Cartesian Unit Vectors
A unit vector is simply a vector whose magnitude is 1.
- A unit vector is a convenient way to specify a direction.
- A unit vector having the same direction as \( \mathbf{A} \) is represented by

\[
\mathbf{u}_A = \frac{\mathbf{A}}{A}
\]

Any vector \( \mathbf{A} \) can be regarded as the product of its magnitude and a unit vector that has the same direction as \( \mathbf{A} \).
- Dividing the vector \( \mathbf{A} \) by its magnitude produces a unit vector that has a magnitude of 1 and the same direction as \( \mathbf{A} \).

In three dimensions, the set of Cartesian unit vectors, \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) is used to designate the directions of the x, y, z-axes, respectively.
- The positive Cartesian unit vectors are shown at the right.

Cartesian Vector Representation
We can write \( \mathbf{A} \) in Cartesian vector form as follows.

\[
\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}
\]

There is a distinct advantage to writing vectors in this manner.
- The magnitude and direction of each component are separated, and this simplifies the operations of vector algebra, particularly in three dimensions.

Magnitude of a Cartesian Vector
The magnitude of \( \mathbf{A} \) may be found using the Pythagorean Theorem.

\[
A = (A_x^2 + A_y^2 + A_z^2)^{\frac{1}{2}}
\]
Direction of a Cartesian Vector
In the three dimensions of space, the direction of \( \mathbf{A} \) is defined by the coordinate direction angles \( \alpha \) (alpha), \( \beta \) (beta), and \( \gamma \) (gamma) measured between the line of action of the vector and the positive x, y, z-axes, respectively.

The direction cosines can be used to determine these angles.
\[
\begin{align*}
\cos \alpha &= \frac{A_x}{A} \\
\cos \beta &= \frac{A_y}{A} \\
\cos \gamma &= \frac{A_z}{A}
\end{align*}
\]

\( \alpha \) = the angle between the x-axis and the line of action of the vector \( \mathbf{A} \).

\( \beta \) = the angle between the y-axis and the line of action of the vector \( \mathbf{A} \).

\( \gamma \) = the angle between the z-axis and the line of action of the vector \( \mathbf{A} \).

The scalar components of vector \( \mathbf{A} \) can be defined using the direction cosines.
\[
\begin{align*}
A_x &= A \cos \alpha \\
A_y &= A \cos \beta \\
A_z &= A \cos \gamma
\end{align*}
\]

Using Cartesian vector notation, vector \( \mathbf{A} \) can be expressed as
\[
\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}
\]

and using the direction cosines, vector \( \mathbf{A} \) can be expressed as
\[
\mathbf{A} = A \left( \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k} \right)
\]

The vector \( \mathbf{A} \) can also be expressed as a scalar \( A \) and a unit vector,
\[
\mathbf{A} = A \mathbf{u}_A
\]

where \( \mathbf{u}_A = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k} \)

\( \mathbf{u}_A \) is the unit vector along the line of action of \( \mathbf{A} \).
An easy way to obtain the direction cosines of \( \mathbf{A} \) is to form a unit vector in the direction of \( \mathbf{A} \):

\[
\mathbf{u}_A = \mathbf{A}/A = (A_x/A) \mathbf{i} + (A_y/A) \mathbf{j} + (A_z/A) \mathbf{k}
\]

where \( A = (A_x^2 + A_y^2 + A_z^2)^{\frac{1}{2}} \)

By comparison,

\[
\mathbf{u}_A = (\cos \alpha) \mathbf{i} + (\cos \beta) \mathbf{j} + (\cos \gamma) \mathbf{k}
\]

Since the magnitude of the unit vector is 1, then

\[
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1
\]

2.6 Addition of Cartesian Vectors

The vector operations of addition and subtraction of two or more vectors are greatly simplified if the vectors are expressed in terms of Cartesian components.

The resultant of two or more forces in space is determined by summing their rectangular components.

\[
\mathbf{R} = \mathbf{A} + \mathbf{B} = (A_x + B_x) \mathbf{i} + (A_y + B_y) \mathbf{j} + (A_z + B_z) \mathbf{k}
\]

If the above concept of vector addition is generalized and applied to a system of several concurrent forces, then the force resultant is the vector sum of all the forces in the system and can be written as

\[
\mathbf{F}_R = \sum \mathbf{F} = \sum (F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}) = \sum F_x \mathbf{i} + \sum F_y \mathbf{j} + \sum F_z \mathbf{k}
\]

The scalar components can be found as follows.

\[
F_{RX} = \sum F_x \quad F_{RY} = \sum F_y \quad F_{RZ} = \sum F_z
\]

Graphical and trigonometric methods are generally not practical in the case of forces in space.
Example Problem - 3D Force Components and Direction Angles

Given: The forces shown.

Find: a) $x$, $y$, and $z$-components
    b) $\alpha$, $\beta$, and $\gamma$

a) $\beta = 90^\circ - 65^\circ = 25^\circ$ \hspace{1cm} Answer

\[ F_y = 900 \cos 25^\circ = 816 \text{ N} \] \hspace{1cm} Answer

\[ F_h = 900 \cos 65^\circ = 380 \text{ N} \]

\[ F_x = -F_h \sin 20^\circ = -130.1 \text{ N} \] \hspace{1cm} Answer

\[ F_z = F_h \cos 20^\circ = 357 \text{ N} \] \hspace{1cm} Answer

b) $\beta = 25^\circ$ \hspace{1cm} Answer (from above)

\[ \cos \beta = \cos 25^\circ = 0.906 \]

\[ \cos \alpha = F_x/F = -130.1/900 = -0.1446 \]

\[ \alpha = 98.3^\circ \] \hspace{1cm} Answer

\[ \cos \gamma = F_z/F = 357/900 = 0.397 \]

\[ \gamma = 66.6^\circ \] \hspace{1cm} Answer

\[ \mathbf{F} = (-130.1 \mathbf{i} + 816 \mathbf{j} + 357 \mathbf{k}) \text{ N} \]

\[ \mathbf{F} = 900 (-0.1446 \mathbf{i} + 0.906 \mathbf{j} + 0.397 \mathbf{k}) \text{ N} \]
2.7 Position Vectors

\textbf{x, y, z Coordinates}

Points in space are located relative to the origin of the coordinates by successive measurements along the x-, y-, and z-axes.

- The result yields coordinates at the required point, such as \((x, y, z)\).

\textbf{Position Vector}

The position vector \(\mathbf{r}\) is defined as a fixed vector that locates a point in space relative to another point.

\[ \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \]

In many applications, the direction of the vector \(\mathbf{r}\) is defined by the coordinates of two points located on its line of action.

\[ \begin{align*}
    dx &= x_B - x_A \quad \text{(head - tail)} \\
    dy &= y_B - y_A \\
    dz &= z_B - z_A
\end{align*} \]

The position vector is determined as follows.

\[ \mathbf{r} = (x_B - x_A) \mathbf{i} + (y_B - y_A) \mathbf{j} + (z_B - z_A) \mathbf{k} \]

and \[ d = (dx^2 + dy^2 + dz^2)^{1/2} \]

2.8 Force Vector Directed Along a Line

Often in three-dimensional problems, the direction of a force is specified by two points through which its line of action passes.

We can formulate \(\mathbf{F}\) as a Cartesian vector using a unit vector in the direction of a position vector \(\mathbf{r}\) that acts in the direction of the force.

- The position vector \(\mathbf{r}\) is determined as follows, with points A and B on the line of action of the force \(\mathbf{F}\).

\[ \mathbf{r} = (x_B - x_A) \mathbf{i} + (y_B - y_A) \mathbf{j} + (z_B - z_A) \mathbf{k} \]
• If the position vector \( \mathbf{r} \) is divided by its magnitude, then a unit vector \( \mathbf{u} \) that has the same direction as the vector \( \mathbf{F} \) can be obtained.

\[
\mathbf{u} = \frac{\mathbf{r}}{|\mathbf{r}|}
\]

• Thus, \( \mathbf{F} \) can be expressed as the product of its magnitude and \( \mathbf{u} \).

\[
\mathbf{F} = F \mathbf{u} = F \left( \frac{\mathbf{r}}{|\mathbf{r}|} \right)
\]

We can also formulate \( \mathbf{F} \) as a Cartesian vector using the components of the force and corresponding unit vectors in the direction of the x, y, and z-axes.

• Rather than determining the components using a position vector, the components of the force may be found as follows (a bit of a “short cut”).

\[
F_x = F \left( \frac{dx}{d} \right)
\]

\[
F_y = F \left( \frac{dy}{d} \right)
\]

\[
F_z = F \left( \frac{dz}{d} \right)
\]

The direction cosines are determined as follows.

\[
\cos \alpha = \frac{dx}{d} = \frac{F_x}{F}
\]

\[
\cos \beta = \frac{dy}{d} = \frac{F_y}{F}
\]

\[
\cos \gamma = \frac{dz}{d} = \frac{F_z}{F}
\]
Example Problem – 3D Resultant

Given: The two forces shown.

Find: The resultant force.

**Force F₁**
\[ \begin{align*}
  dx &= 4 - 2 = 2 \\
  dy &= 8 - 4 = 4 \\
  dz &= -12 - (-6) = -6 \\
  d &= 7.48 \\
  F_{1x} &= F_1 \left(\frac{dx}{d}\right) = 100 \left(\frac{2}{7.48}\right) = 26.7 \text{ lb} \\
  F_{1y} &= F_1 \left(\frac{dy}{d}\right) = 100 \left(\frac{4}{7.48}\right) = 53.5 \text{ lb} \\
  F_{1z} &= F_1 \left(\frac{dz}{d}\right) = 100 \left(-\frac{6}{7.48}\right) = -80.2 \text{ lb}
\end{align*} \]

**Force F₂**
\[ \begin{align*}
  dx &= 2 - 10 = -8 \\
  dy &= 4 - 6 = -2 \\
  dz &= -6 - 6 = -12 \\
  d &= 14.56 \\
  F_{2x} &= F_2 \left(\frac{dx}{d}\right) = 200 \left(-\frac{8}{14.56}\right) = -109.9 \text{ lb} \\
  F_{2y} &= F_2 \left(\frac{dy}{d}\right) = 200 \left(-\frac{2}{14.56}\right) = 27.5 \text{ lb} \\
  F_{2z} &= F_2 \left(\frac{dz}{d}\right) = 200 \left(-\frac{12}{14.56}\right) = 164.8 \text{ lb}
\end{align*} \]

The resultant
\[ \begin{align*}
  R_x &= 26.7 - 109.9 = -83.2 \text{ lb} \\
  R_y &= 53.5 - 27.5 = 26.0 \text{ lb} \\
  R_z &= -80.2 - 164.8 = -245.0 \text{ lb} \\
  R &= 260.0 \text{ lb}
\end{align*} \]

Direction cosines and direction angles for the resultant are
\[ \begin{align*}
  \cos \alpha &= \frac{-83.2}{260.0} = -0.320 \quad \therefore \alpha = 108.7^\circ \\
  \cos \beta &= \frac{26.0}{260.0} = +0.100 \quad \therefore \beta = 84.3^\circ \\
  \cos \gamma &= \frac{-245.0}{260.0} = -0.942 \quad \therefore \gamma = 160.4^\circ
\end{align*} \]

Vector notations for the resultant are
\[ \begin{align*}
  \boldsymbol{R} &= (-83.2 \hat{i} + 26.0 \hat{j} - 245.0 \hat{k}) \text{ lb} \\
  \boldsymbol{R} &= 260.0 (-0.320 \hat{i} + 0.100 \hat{j} - 0.942 \hat{k}) \text{ lb}
\end{align*} \]
2.9 Dot Product

The scalar product of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) is defined as the product of the magnitudes of \( \mathbf{A} \) and \( \mathbf{B} \) and of the cosine of the angle \( \theta \) formed by \( \mathbf{A} \) and \( \mathbf{B} \).

\[
\mathbf{A} \cdot \mathbf{B} = AB \cos \theta
\]

\( \mathbf{A} \cdot \mathbf{B} \) is also referred to as the "dot product" of the vectors \( \mathbf{A} \) and \( \mathbf{B} \).

- The result of the dot product operation is a scalar.

Laws of Operation
1. Commutative law: \( \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \)
2. Multiplication by a scalar: \( a (\mathbf{A} \cdot \mathbf{B}) = (a \mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (a \mathbf{B}) \)
3. Distributive law: \( \mathbf{A} \cdot (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \cdot \mathbf{D}) \)

Cartesian Vector Formulation

\[
\begin{align*}
\mathbf{i} \cdot \mathbf{i} &= 1 & \mathbf{i} \cdot \mathbf{j} &= 0 & \mathbf{i} \cdot \mathbf{k} &= 0 \\
\mathbf{j} \cdot \mathbf{i} &= 0 & \mathbf{j} \cdot \mathbf{j} &= 1 & \mathbf{j} \cdot \mathbf{k} &= 0 \\
\mathbf{k} \cdot \mathbf{i} &= 0 & \mathbf{k} \cdot \mathbf{j} &= 0 & \mathbf{k} \cdot \mathbf{k} &= 1
\end{align*}
\]

The dot product of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) in terms of their components is

\[
\mathbf{A} \cdot \mathbf{B} = (A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}) \cdot (B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k})
= A_x B_x (\mathbf{i} \cdot \mathbf{i}) + A_x B_y (\mathbf{i} \cdot \mathbf{j}) + A_x B_z (\mathbf{i} \cdot \mathbf{k})
+ A_y B_x (\mathbf{j} \cdot \mathbf{i}) + A_y B_y (\mathbf{j} \cdot \mathbf{j}) + A_y B_z (\mathbf{j} \cdot \mathbf{k})
+ A_z B_x (\mathbf{k} \cdot \mathbf{i}) + A_z B_y (\mathbf{k} \cdot \mathbf{j}) + A_z B_z (\mathbf{k} \cdot \mathbf{k})
\]

\[
\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z
\]

Applications
1. Determine the angle formed between two vectors or intersecting lines.

\[
\cos \theta = (\mathbf{A} \cdot \mathbf{B})/AB
\]

2. Determine the components of a vector parallel and perpendicular to a line.

- The component of a vector parallel to a line is called the projection of the vector onto the line.

We can determine the components, \( \mathbf{A}_p \) and \( \mathbf{A}_n \), of a vector parallel and normal to a line using the dot product.
The Parallel Component
In terms of the angle $\theta$ between $\mathbf{A}$ and the component $\mathbf{A}_p$, the magnitude of $\mathbf{A}_p$ is

$$|\mathbf{A}_p| = |\mathbf{A}| \cos \theta$$

Let $\mathbf{u}$ be a unit vector parallel to the line $L$.

- The dot product of $\mathbf{u}$ and $\mathbf{A}$ is

$$\mathbf{u} \cdot \mathbf{A} = |\mathbf{u}| \ |\mathbf{A}| \cos \theta = |\mathbf{A}| \cos \theta$$

Compare this result with the magnitude of $\mathbf{A}_p$ found above.

Therefore the parallel component, or the projection of $\mathbf{A}$ onto line $L$, is

$$\mathbf{A}_p = (\mathbf{u} \cdot \mathbf{A}) \mathbf{u}$$

The Normal Component
Once the parallel component has been determined, the normal component can be found from the relation $\mathbf{A} = \mathbf{A}_p + \mathbf{A}_n$

$$\mathbf{A}_n = \mathbf{A} - \mathbf{A}_p$$
Example Problem – Dot Product

Given: \( \mathbf{A} = -4 \mathbf{i} + 6 \mathbf{j} - 7 \mathbf{k} \) and \( \mathbf{B} = 3 \mathbf{i} - 2 \mathbf{j} + \mathbf{k} \)

Find: Components of \( \mathbf{B} \) parallel and perpendicular to the line of action of \( \mathbf{A} \) (i.e. \( \mathbf{B}_p \) and \( \mathbf{B}_n \)).

First, find the unit vector in the direction of \( \mathbf{A} \) (i.e. \( \mathbf{u} \)).

\[
|\mathbf{A}| = \sqrt{(-4)^2 + (6)^2 + (-7)^2} = 10.05
\]

Therefore,

\[
\mathbf{u} = \frac{1}{10.05} (-4 \mathbf{i} + 6 \mathbf{j} - 7 \mathbf{k})
\]

\[
\mathbf{u} = (-0.398 \mathbf{i} + 0.597 \mathbf{j} - 0.697 \mathbf{k})
\]

Find the component of \( \mathbf{B} \) parallel to the line of action of \( \mathbf{A} \) (i.e. \( \mathbf{B}_p \)).

\[
|\mathbf{B}_p| = |\mathbf{B} \cdot \mathbf{u}| = (3 \mathbf{i} - 2 \mathbf{j} + \mathbf{k}) \cdot (-0.398 \mathbf{i} + 0.597 \mathbf{j} - 0.697 \mathbf{k})
\]

\[
= (3)(-0.398) + (-2)(0.597) + 1(-0.697)
\]

\[
= -1.194 - 1.194 - 0.697 = -3.085
\]

\[
\mathbf{B}_p = |\mathbf{B}_p| (\mathbf{u}) = -3.085 (-0.398 \mathbf{i} + 0.597 \mathbf{j} - 0.697 \mathbf{k})
\]

Thus,

\[
\mathbf{B}_p = +1.228 \mathbf{i} - 1.842 \mathbf{j} + 2.150 \mathbf{k}
\]

Find the component of \( \mathbf{B} \) perpendicular to the line of action of \( \mathbf{A} \) (i.e. \( \mathbf{B}_n \)).

Since \( \mathbf{B} = \mathbf{B}_p + \mathbf{B}_n \),

\[
\mathbf{B}_n = \mathbf{B} - \mathbf{B}_p = (3 \mathbf{i} - 2 \mathbf{j} + \mathbf{k}) - (+1.228 \mathbf{i} - 1.842 \mathbf{j} + 2.150 \mathbf{k})
\]

\[
= (3 - 1.228) \mathbf{i} + (-2 + 1.842) \mathbf{j} + (1 - 2.150) \mathbf{k}
\]

Thus,

\[
\mathbf{B}_n = 1.772 \mathbf{i} - 0.158 \mathbf{j} - 1.150 \mathbf{k}
\]

Check: \( \mathbf{B}^2 = \mathbf{B}_p^2 + \mathbf{B}_n^2 \)

\[
\mathbf{B} = [(3)^2 + (-2)^2 + (1)^2]^{\frac{1}{2}} = 3.742
\]

\[
\mathbf{B}_p = -3.085 \text{ (from above)}
\]

\[
\mathbf{B}_n = [(1.772)^2 + (-0.158)^2 + (-1.150)^2]^{\frac{1}{2}} = 2.118
\]

\[
\mathbf{B} = [(-3.085)^2 + 2.118^2]^{\frac{1}{2}} = 3.742
\]

Check angle \( \theta \) between \( \mathbf{B}_p \) and \( \mathbf{B}_n \)

\[
\mathbf{B}_p \cdot \mathbf{B}_n = |\mathbf{B}_p| |\mathbf{B}_n| \cos \theta
\]

\[
\mathbf{B}_p \cdot \mathbf{B}_n = (+1.228 \mathbf{i} - 1.842 \mathbf{j} + 2.150 \mathbf{k}) \cdot (1.772 \mathbf{i} - 0.158 \mathbf{j} - 1.150 \mathbf{k})
\]

\[
= (1.228)(1.772) + (-1.842)(-0.158) + (2.150)(-1.150)
\]

\[
= 2.176 + 0.291 - 2.473 = -0.006
\]

\[
\cos \theta = \frac{(\mathbf{B}_p \cdot \mathbf{B}_n)}{|\mathbf{B}_p| |\mathbf{B}_n|} = -0.006/(-3.085)(2.118) = 0.00092
\]

and \( \theta = 90.0^\circ \)